Adventures of the Coupled Yang-Mills Oscillators: I. Semiclassical Expansion

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Abstract. We study the quantum mechanical motion in the x^2y^2 potentials with $n = 2, 3$, which arise in the spatially homogeneous limit of the Yang-Mills (YM) equations. These systems show strong stochasticity in the classical limit ($\hbar = 0$) and exhibit a quantum mechanical confinement feature. We calculate the partition function $Z(t)$ going beyond the Thomas-Fermi (TF) approximation by means of the semiclassical expansion using the Wigner-Kirkwood (WK) method. We derive a novel compact form of the differential equation for the WK function. After separating the motion in the channels of the equipotential surface from the motion in the central region, we show that the leading higher-order corrections to the TF term vanish up to eighth order in $\hbar$, if we treat the quantum motion in the hyperbolic channels correctly by adiabatic separation of the degrees of freedom. Finally, we obtain an asymptotic expansion of the partition function in terms of the parameter $g^2\hbar^4t^3$.

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1. Introduction

The discovery of the chaoticity of the classical Yang-Mills (YM) equations [1] (see [2] for a review) has attracted broad interest to the system of two (three) coupled quartic oscillators with the potential x^2y^2 $(x^2y^2 + y^2z^2 + z^2x^2)$, where $x, y\,(z)$ are functions of time t. These systems are the simplest limiting cases for the homogeneous YM equations (sometimes called YM classical mechanics) with $n = 2$ $(n = 3)$ degrees of freedom, respectively.The x^2y^2 model, the central object of the present paper, exhibits a rich chaotic behavior despite its extreme simplicity. Not surprisingly, this model has been encountered in various fields of science, including chemistry, astronomy, astrophysics, and cosmology (chaotic inflation).

Quantum mechanically, this model (YM quantum mechanics, YMQM), despite possessing a logarithmically divergent volume of energetically accessible phase space [2, 3] (we set $m = 1$ throughout) ‡

$$\Gamma_E = \int_{-\infty}^{\infty} dxdydp_xdp_y\,\delta\left(\frac{1}{2}(p_x^2 + p_y^2) + \frac{g^2}{2}x^2y^2 - E\right), \tag{1}$$

can be shown to have a discrete spectrum [5, 6]. Physically, it is clear why this is so: Quantum fluctuations, e.g. zero-point fluctuations, forbid the "particle" to escape along the x or y axis where the potential energy vanishes. The system is thus confined to a finite volume, and this implies the discreteness of the energy levels. Classically, of course, the particle can always escape along one of the axes without increasing its energy.

In this article we calculate the partition function (heat kernel) $Z(t)$ for the YMQM with the above given potentials beyond the well-known Thomas-Fermi (TF) approximation, which takes into account only the discreteness of the quantum mechanical phase space, but treats the Hamiltonian classically. There are several interesting articles [7, 8] devoted to the approximate calculation of $Z(t)$ in the TF approximation for these potentials. They are based on the adiabatic separation of the dependence of Z on x and y in the narrow channels of the equipotential surface $xy = \text{const.}$. The range of the integration over the coordinates x, y and the momenta p_x, p_y was divided into two regions: the central region $(|x|, |y| \leq Q)$ and the channels $(Q \leq |x|, Q \leq |y|)$, which are governed by quite different physics: The system is essentially classical in the central region and intrinsically quantum mechanical in the channels performing oscillatory motion with x-dependent frequency in the "fast" variable y (in the channels along the x-axis), but quasi-free motion in the "slow" variable x.

The dependence on the artificial boundary Q dividing the central region from the channels disappears in the final answer for $Z(t)$. Below we show that this property survives beyond the TF approximation up to the order $\hbar^8$. In the paper [9] an alternative to this approach of calculating $Z(t)$ was proposed, starting from the Yang-Mills-Higgs

‡ This is in violation of Weil's famous theorem, which states that the average number $N(E)$ of energy levels with energy less than E is asymptotically proportional to $\int_0^E \Gamma_{E'} dE'$.

quantum mechanics (YMHQM) and then passing to the limit $v = 0$, where v is the vacuum expectation value of the "Higgs field" defining the strength of the harmonic potential.

Here we study the x^2y^2 potential and postpone the investigation of YMHQM to a separate publication [10]. We here follow the method of separation of the domain of motion into two regions, the central square $x, y \in [-Q, Q]$ and the hyperbolic channels, introduced by Tomsovic [7] and Whelan [8]. The motion in the channels can be treated by adiabatic separation of the motion in the slow variable (in the direction of the channel) and the motion in the fast variable (perpendicular to the channel). We will go beyond the Thomas-Fermi approximation used in [7, 8] by applying the well-known Wigner-Kirkwood (WK) method [11, 12, 13] (see [14] for a review of the WK approach).

We already mentioned that the motion in the central region, where the variables x and y are treated on an equal footing, is quite different from the motion in the channels, where the motion in the perpendicular direction (for now taken as the variable y) performs quantum oscillations with x-dependent frequency, but the motion in the x variable is not treated as free as it was done in [7, 8]. We apply the WK method in the central region to both variables, but only to the slow variable x in the channels treating the motion in the fast y variable fully quantum mechanically. We improve the method of adiabatic separation introduced in [7, 8] by taking into account the effect of the x-dependence of the oscillation frequency onto the motion in the x-direction. This is especially important, as we shall see, for the higher-order quantum corrections. We show that, up to eighth-order in $\hbar$ and in the leading terms in $(tQ^4)^{-1} \ll 1$, the calculated partition function $Z(t)$ does not depend on the boundary Q dividing the two regions and the quantum corrections from both regions cancel.

An interesting phenomenon occurs in the channel motion due to the presence of terms suppressed by powers of $(tQ^4)^{-1}$, which are independent of Q. Each term of this kind is negligible in comparison with the leading (but canceling) terms, but their series forms an asymptotic series in the variable $g^2\hbar^4t^3$, which does not involve Q. If we postulate that the entire Q-dependence of the partition function disappears, which we prove for the leading terms in $(tQ^4)^{-1}$, only this series will be left as the contribution from the channels. We may express this phenomenon as the "transmutation" of the small expansion parameter $(tQ^4)^{-1}$ governing the approach of adiabatic separation of variables into the small parameter $g^2\hbar^4t^3$ characteristic of quantum mechanics. The quasiclassical expansion is shown to have the nature of an asymptotic series.

In the next two sections we present the YMQM system and the WK method of calculating $Z(t)$ beyond the TF approximation.

2. Yang-Mills-Higgs classical and quantum mechanics: The Thomas-Fermi approximation

For spatially homogeneous fields (long wave length limit of the Yang-Mills field) the classical Hamiltonian for $n = 2$ is given by the expression

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{g^2}{2}x^2y^2. \tag{2}$$

The quantized counterpart of (2) is

$$\hat{H} = -\frac{\hbar^2}{2}\nabla^2_{x,y} + \frac{g^2}{2}x^2y^2. \tag{3}$$

A brief word on units: All quantities are given below in units of the energy E with dimensions $[H] = 1, [t] = -1, [x], [y] = 1/4, [g] = 0, [\hbar] = 3/4$. The operator (3) has a discrete spectrum [5, 6]. The TF approximation to the heat kernel or partition function $Z(t) = \mathrm{Tr}[\exp(-t\hat{H})]$ is the standard lowest-order semiclassical approximation valid for small $\hbar t^{3/4} \ll 1$. It is obtained by substituting the classical Hamiltonian for its quantum counterpart and replacing the trace of the heat kernel by the integral over the phase-space volume normalized by $(2\pi\hbar)^{-n}$, where $2n$ is the phase-space dimension. In other words, the TF approximation takes into account only the discreteness of the quantum mechanical phase space, but considers momenta and coordinates (in our case, the field amplitudes x and y) as commuting variables. This method was used in numerous papers (see e.g. [7, 8, 9]). For the calculation of the energy level density $\rho(E) = dN(E)/dE$ at asymptotic energies, the TF approximation is a consistent approach since, as we shall see below, all corrections to the TF term are structures with factors $\hbar^k t^\ell$ with k, ℓ positive integers. For the asymptotic energy level density $\rho(E)$ or $N(E)$ these corrections are negligible according the Karamata-Tauberian theorems [5, 6] relating the most singular part of $Z(t)$ to the asymptotic level density: $N(E) = \int dE\rho(E) = L^{-1}(Z(t)/t)$ where L^{-1} denotes the inverse Laplace transform.

For the Hamiltonian (3) the naive TF approximation is divergent, because the classical phase space is infinite. As explaines in the Introduction, finite results for $Z(t)$ and $N(E)$ can be obtained by including certain quantum corrections to the channel motion by means of the method of adiabatic separation of the motion along and perpendicular to the channels [7, 8]. We here give the expression for the partition function obtained in this improved TF approximation in our notation:

$$Z_0 = \frac{1}{\sqrt{2\pi}g\hbar^2 t^{3/2}}\left(\ln\frac{1}{g^2\hbar^4 t^3} + 9\ln 2 + C\right), \tag{4}$$

where C is the Euler constant. We shall return to (4) again below. Because we shall often encounter the pre-factor appearing in (4), we introduce the special symbol for it:

$$K \equiv (2\pi g^2\hbar^4 t^3)^{-1/2}. \tag{5}$$

3. Beyond the TF approximation: The Wigner-Kirkwood expansion

For non asymptotic energies and also for the fluctuating part of the level density one needs to go beyond the TF approximation and calculate the quantum corrections to the TF term in $Z(t)$. We remark that this problem is interesting not only from this practical point of view, but also because it provides insight into some problems of perturbation theory in quantum mechanics. Here we use the Wigner-Kirkwood (WK) method. Before we proceed, we emphasize the dual role of the variables $x(t), y(t), z(t)$. They formally play the role of the coordinates, but at the same time they stand for the homogeneous gauge field amplitudes. The trace in the heat kernel may be taken with respect to any complete set of states. For the semiclassical expansion involving integration over the phase space for the quantum operators in the Wigner representation it is convenient to use the plane waves as a complete set:

$$Z(t) = \frac{1}{(2\pi\hbar)^n} \int \prod_{i=1}^{n} dx_i dp_i \, e^{-i\vec{p}\vec{r}/\hbar} e^{-t\hat{H}} e^{i\vec{p}\vec{r}/\hbar} \tag{6}$$

with $\vec{r} = (x_1, x_2, \ldots, x_n), \vec{p} = (p_1, p_2, \ldots, p_n)$. The kinetic energy term in the exponent of $\exp(-t\hat{H})$ does not commute with the potential energy $V(\vec{x})$; the WK expansion provides a convenient method of calculating the noncommuting terms. Following [13] we set

$$e^{-t\hat{H}} e^{i\vec{p}\vec{r}/\hbar} = e^{-tH} e^{i\vec{p}\vec{r}/\hbar} W(\vec{r}, \vec{p}; t) = u(\vec{r}, \vec{p}; t) \tag{7}$$

where the function $W(\vec{r}, \vec{p}; t)$ is to be determined. $H(\vec{p}, \vec{r})$ (as opposed to $\hat{H}$) is the classical Hamiltonian (2). The function $u(\vec{r}, \vec{p}; t)$ satisfies the Bloch equation:

$$\frac{\partial u}{\partial t} + \hat{H} u = 0 \tag{8}$$

with the boundary condition

$$\lim_{t \to 0} u(\vec{r}, \vec{p}; t) = e^{i\vec{p}\vec{r}/\hbar}, \tag{9}$$

corresponding to the initial condition $W(\vec{r}, \vec{p}; 0) = 1$. From (7) and (8) we obtain an exact equation for W:

$$\begin{aligned} \frac{\partial W}{\partial t} = \frac{\hbar^2}{2} \Big[& \Delta - t(\Delta V) - \frac{2it}{\hbar} (\vec{p} \cdot \nabla V) + t^2 (\nabla V)^2 \\ & + \frac{2}{\hbar} (i\vec{p} - \hbar t \nabla V) \cdot \nabla \Big] W, \end{aligned} \tag{10}$$

where $\Delta = \nabla^2$ is the Laplacian. Next we expand W in powers of $\hbar$:

$$W = \sum_{k=0}^{\infty} \hbar^k W_k \tag{11}$$

and equate the terms with the same power of $\hbar$ on both sides. We thus obtain a recurrence relation of differential equations for the W_k:

$$\begin{aligned} \frac{\partial W_k}{\partial t} = & \frac{1}{2} \Big[\Delta - t(\Delta V) + t^2 (\nabla V)^2 - 2t \nabla V \cdot \nabla \Big] W_{k-2} \\ & + i\vec{p} \cdot [\nabla - t(\nabla V)] W_{k-1}, \end{aligned} \tag{12}$$

with the initial conditions $W_k = 0$ for $k < 0, W_0 = 1$. Since $Z(t)$ and W are linearly related, the expansion of W in powers of $\hbar$ immediately translates into an expansion of $Z(t)$ in powers of $\hbar$. To obtain the term $Z_k(t)$, one needs to calculate W_k from (12) and integrate over $\vec{p}$ and $\vec{x}$:

$$Z_k(t) = \frac{\hbar^k}{(2\pi\hbar)^n} \int_{-\infty}^{\infty} \prod_{i=1}^{n} dx_i dp_i W_k(\vec{r}, \vec{p}; t) \exp\left[-t\left(\frac{\vec{p}^{\,2}}{2} + V(\vec{x})\right)\right]. \tag{13}$$

The expressions (10) and (12) may be written in more compact form, if we introduce the "covariant derivative" $\vec{D} \equiv \nabla - t\nabla V$:

$$\frac{\partial W}{\partial t} = \frac{\hbar^2}{2}\left[D^2 + \frac{2i}{\hbar}\vec{p}\cdot\vec{D}\right] W = \frac{1}{2}\left[(\hbar\vec{D} + i\vec{p})^2 + \vec{p}^{\,2}\right] W \tag{14}$$

or its recursive form

$$\frac{\partial W_k}{\partial t} = \frac{1}{2}\left[D^2 W_{k-2} + 2i\vec{p}\cdot\vec{D}W_{k-1}\right]. \tag{15}$$

We emphasize that the symbol $\vec{p}$ here denotes a classical phase-space variable and not an operator. As far as we know, the form (14) of the equation (10) has not previously been presented in the literature.

In terms of the operator $\vec{D}$, eq. (14) resembles a Fokker-Planck equation for $W(\vec{r}, \vec{p}; t)$ with the diffusion constant $\sigma = \hbar^2$ and the drift vector $\vec{\gamma} = -i\hbar\vec{p}$ (see text below eq. (16)). The relation to the Fokker-Planck equation can be further elucidated by noting that the "vector potential" $\vec{A} = t\nabla V$ is a complete gradient and thus can be "gauged" away by means of the transformation $W \to e^{tV}W'$, yielding an alternative form of (14):

$$\frac{\partial W'}{\partial t} = \frac{\hbar^2}{2}\left[\nabla^2 + \frac{2i}{\hbar}\vec{p}\cdot\nabla\right] W' - VW'. \tag{16}$$

If we interpret $W'(\vec{r}, \vec{p}; t)$ as a one-time probability density and introduce the probability current (sometimes called the probability flux in the literature) [15]

$$\vec{J}' = -i\hbar\vec{p}W' - \frac{\hbar^2}{2}\nabla W', \tag{17}$$

we may write (16) in the form of a continuity equation:

$$\frac{\partial W'}{\partial t} + \nabla\cdot\vec{J}' = -VW', \tag{18}$$

where the potential term acts as a source term and violates the local conservation law associated with the Fokker-Planck equation.

The recurrence relation (15) clearly shows that the expansion in $\hbar$ introduced by Kirkwood [12] is also an expansion in powers of the gradient operator as emphasized by Uhlenbeck and Beth [13]. In the general case one needs to expand in powers of $\hbar$ or, equivalently, in powers of the gradient operator using (12) or (15), as we will do here. In special cases, however, the compact form of the equations (14) or (16) may be the starting point of other effective approximation schemes.

4. Testing the method of separation beyond the TF approximation for the x^2y^2 potential

We now want to check whether the method of the separation x- and y- motions in the channels [7, 8] works beyond the TF approximation. In the TF approximation, where the integrand in $Z(t)$ is simply $\exp(-tH)$, the boundary Q appears in the argument of a logarithm. Higher WK corrections to the TF term lead to a power-like dependence on Q, and it is important to confirm that all dependence on Q is cancelled in the final answer, at least for the leading terms in the parameter $tQ^4 \gg 1$. Here we consider this problem up to the second-order corrections. In the later sections we will consider corrections up to the order $\hbar^8$.

As we stressed before, we apply the WK method to the motion in the central region ($|x|, |y| \leq Q$) in its full scope, whereas for the motion in the channels ($|x|, |y| \geq Q$) this method will be used only to study the quantum effects on the "slow" longitudinal motion, which is adiabatically separated from the quantized oscillatory motion in the transverse coordinate.

Integrating (12) with respect to t, we have:

$$\begin{aligned} W_1 &= -\frac{it}{2}\vec{p}\cdot\nabla V \\ W_2 &= \frac{t^2}{2}\left[-\frac{1}{2}\Delta V + \frac{t}{3}(\nabla V)^2 + \frac{t}{3}\sum_{i,k} p_i p_k \frac{\partial^2 V}{\partial x_i \partial x_k} - \frac{t^2}{4}(\vec{p}\cdot\nabla V)^2\right] \end{aligned} \tag{19}$$

with $V(x,y) = \frac{1}{2}g^2x^2y^2$. Integration over p_x, p_y and making use of the symmetry of the Hamiltonian with respect to the interchange $x \leftrightarrow y$, we find that the contribution from W_1 vanishes and

$$\int d\Gamma\, W_2\, e^{-tV} = \frac{\pi t g^2}{3}\left[-I_{10} + \frac{tg^2}{2}I_{21}\right], \tag{20}$$

where we introduced the abbreviations

$$d\Gamma = dx dy dp_x dp_y \tag{21}$$

and (for $m \geq n$)§:

$$I_{mn} = 4\int_0^Q dx \int_0^Q dy\, x^{2m}y^{2n}e^{-tg^2x^2y^2/2}. \tag{22}$$

These integrals can be evaluated straightforwardly after the substitution $x = w$ and $y = \sqrt{2/t}(u/gw)$ and with the help of the condition for the validity of the adiabatic approximation $Qt^{1/4} \gg 1$. Note that this inequality does not contradict the condition permitting the use of the Wigner representation $\hbar Qt \ll 1$ if $\hbar t^{3/4} \ll 1$. We obtain

$$I_{10} \approx \frac{\sqrt{2\pi}}{gt^{1/2}}Q^2, \qquad I_{21} \approx \frac{\sqrt{2\pi}}{g^3t^{3/2}}Q^2. \tag{23}$$

§ Note that the case $m = n$ needs to be calculated separately from the case $m > n$ for the leading terms; see below.

and finally for $Z_2(t)$ in the square:

$$Z_2 = -K\frac{1}{12}(g\hbar tQ)^2. \tag{24}$$

For the sake of completeness and later use, we note the systematic structure of the integrals I_{mn}. For a given value of $m-n$ there are $m-n+1$ such expressions: $I_{m-n,0}, I_{m-n+1,1}, \ldots, I_{2(m-n),m-n}$. Applying the same method as above yields the following result for these expressions at fixed $m-n$:

$$I_{mn} = \frac{\sqrt{2\pi}}{(gt^{1/2})^{2n+1}}\frac{(2n-1)!!}{m-n}Q^{2(m-n)}. \tag{25}$$

For the second region (the four channels $|x| \in [Q,\infty]$ or $|y| \in [Q,\infty]$), the integration is more involved. WK corrections to the TF term introduce pre-exponential functions of p_x, p_y, x, y and t in W_2. Because of the fourfold symmetry of the Hamiltonian (3) it is sufficient to consider the channel $x \geq Q$. Closely following the method used in [7, 8] we consider the motion in the x variable as free and the motion in the y variable as that of a harmonic oscillator with an x-dependent frequency with the Hamiltonian

$$H_y = \frac{1}{2}p_y^2 + \frac{1}{2}\omega_x^2 y^2 \tag{26}$$

with $\omega_x = gx$ and eigenvalues $e_n(x) = (n+\frac{1}{2})\hbar g x$.

In order to reduce the integrals over p_y^2 and y^2 to derivatives of the well-known exponentiated sum-rule for the harmonic oscillator, we rescale the kinetic energy term in H_y by a factor b and the potential energy term by a factor a. We denote the rescaled Hamiltonians by H'_y. We then can generate any pre-exponential powers of p_y^2 and y^2 by differentiating with respect to a and b and setting $a = b = 1$ in the final result. Integrating W_2 from (19) over p_x and x and performing the described substitutions and differentiations we obtain :

$$\int_{-\infty}^{\infty} dp_x \int_Q^{\infty} dx\, W_2\, \mathrm{Tr}\left(e^{-tH'_y}\right) = 2\sqrt{\frac{2\pi}{t}}(gt)^2 \int_Q^{\infty} dx D(x) \tag{27}$$

with

$$D(x) = \left[-\frac{x^2}{2} - \frac{2x^2}{3}\frac{d}{da} + \frac{1}{3g^2tx^2}\frac{d^2}{da^2} \right.$$
$$\left. -\frac{2x^2}{3}\frac{d}{db} + x^2\frac{d^2}{dadb}\right]\mathrm{Tr}\left(e^{-tH'_y}\right)_{a=b=1}. \tag{28}$$

In (27) we may discard the third term as it is $\mathcal{O}(1/x^4t) \ll 1$ with respect to other terms ($x > Q, Q^4t \gg 1$ in the channel).‖ We note that the algebraic trick dealing with terms including the "fast" variables y, p_y effectively reduce the problem in the channels to the level of TF terms with the rescaled Hamiltonians H'_y. Now we can use the exponentiated

‖ One may notice that contributions arising from the derivative with respect to the "fast" variable y in W_2 are not small, whereas derivatives with respect to x are kinematically negligible in the channel where $Q^4t \gg 1$.

sum-rule for the harmonic oscillator (see [7]) to obtain

$$D(x) = \left[-\frac{x^4}{4\sinh\xi} - \frac{2x^2}{3}\frac{d}{da}\frac{1}{\sinh(\xi\sqrt{a})} \right.$$
$$\left. -\frac{x^2}{2}\frac{d^2}{dadb}\frac{1}{\sinh(\xi\sqrt{ab})} \right]_{a=b=1} . \tag{29}$$

where $\xi = \hbar gtx/2$. Performing the differentiations, setting $a = b = 1$, and multiplying by 4 to account for all four channels, we obtain for the channel contribution to $Z_2(t)$:

$$Z_2(t) = 4K \int_{\xi_0}^{\infty} \xi^2 d\xi \left[-\frac{1}{\sinh\xi} + \frac{11}{6}\xi\frac{\cosh\xi}{\sinh^2\xi} \right.$$
$$\left. +\frac{\xi^2}{2\sinh\xi} - \frac{\xi^2\cosh\xi}{\sinh^3\xi} \right] , \tag{30}$$

with $\xi_0 = \hbar gtQ/2(\ll 1)$. Performing the integrations in (30) (see [16], integrals 2.477.3, 3.523.1, 2.479.4, 2.477.1, 3.523.1), we obtain for the channel contribution to Z_2:

$$Z_2(t) = K\left[\frac{1}{12}(g\hbar tQ)^2 - 21\zeta(3)\right] . \tag{31}$$

where $\zeta(z)$ is the Riemann zeta function ($\zeta(3) \approx 1.202$). From (24) and (31) one sees that the Q-dependence of $Z_2(t)$ cancels, and the final answer is

$$Z_2(t) \approx -25.2K. \tag{32}$$

We achieved the Q-independence of the second-order correction to the partition function of x^2y^2 model using the method of [7, 8]. The huge renormalization of the TF term in (32) looks suspicious.

5. Improved quantum motion in the channels

The unexpectedly large coefficient in (31) suggests that we need to find a better treatment for the motion in the channel. The previous study of the quantum motion in the channel neglected the quantum nature of the "slow" motion along the channel axis, which was assumed to be free in [7, 8]. We already stressed in the Introduction that this motion is by no means free, rather, it is influenced by an effective linear potential created by the quantum fluctuations in the direction(s) orthogonal to the channel axis. Indeed, in the region $|x| \gg |y|$, where the derivatives with respect to x are small relative to the ones with respect to y, we may first average the motion over the quantum fluctuations of y [17] described by the Hamiltonian H_y (26) and by the corresponding wave function

$$\psi_n(y) = \frac{1}{\sqrt{2^n n!}}\left(\frac{gx}{\pi\hbar}\right)^{1/4} e^{-gxy^2/2\hbar} H_n(y\sqrt{gx/\hbar}), \tag{33}$$

where $H_n(z)$ are the Hermite polynomials. The corresponding average value of H_y

$$\langle n|H_y|n\rangle = (n + \frac{1}{2})\hbar gx \tag{34}$$

then becomes an effective potential for the description of the motion in the "slow" variable x:

$$\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2}+(n+\frac{1}{2})\hbar g x\right)\phi_n(x)=E\phi_n(x). \tag{35}$$

This is the well-known Schrödinger equation for a linear potential with solutions in terms of Airy functions. Equation (35) clearly shows that, quantum mechanically, the "particle" is linearly confined along the channel axis and its motion is not free.¶ The eigenvalue problem (35) has a discrete spectrum. Note that this argument constitutes a sixth proof, in addition to the five proofs listed in [5], for the discreteness of the spectrum of the Hamiltonian (3). It explains the large number in (31) and (32) as an artefact of the assumption of free motion in the x-direction. This assumption becomes increasingly poor for the higher-order quantum corrections, leading to poor convergence or even divergence of the expansion in powers of $\hbar$. Treating the "slow" x-motion in the channel adiabatically, we apply the WK expansion and eq. (12) to the effective Hamiltonian

$$H_x^{(n)}=\frac{1}{2}p_x^2+(n+\frac{1}{2})\hbar g x \tag{36}$$

describing the motion of the nth quantum mode in the region $x\geq Q$. For the partition function of the one-dimensional motion of the nth mode we have (denoting p_x simply by p):

$$Z^{(n)}(t)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dp\int_Q^{\infty}dx\,e^{-\frac{1}{2}p^2t-(n+\frac{1}{2})\hbar g x t}W(x,p;t). \tag{37}$$

Expanding $W(x,p;t)$ in powers of $\hbar$ for the linear potential in (36) and using eq. (12), we easily obtain:

$$\begin{aligned}
W_2&=\frac{1}{3\cdot 2^3}a_n^2t^3(4-3p^2t); \\
W_4&=\frac{1}{3^2\cdot 2^7}a_n^4t^6(16-24p^2t+3p^4t^2) \\
W_6&=\frac{1}{2^{10}\cdot 3^3\cdot 5!!}a_n^6t^9(320-720p^2t+180p^4t^2-9p^6t^3), \\
W_8&=\frac{1}{2^{15}\cdot 3^4\cdot 7!!}a_n^8t^{12}(8960-26880p^2t+10080p^4t^2-1008p^6t^3+27p^8t^4),
\end{aligned} \tag{38}$$

where $a_n=(n+\frac{1}{2})\hbar g$. The terms with odd indices contain odd powers of p and give vanishing contributions after integration over p. Carrying out the integration over p in (37), summing over n, and retaining all terms with derivatives up to order $\hbar^8$, we obtain

¶ Note that the phenomenon called "confinement" here is not the same as the phenomenon commonly referred to as quark confinement. In our case, the potential depends linearly on the field amplitude $x(t)$, not on a spatial coordinate. We already emphasized the dual role of the coordinates x, y, z earlier, but we stress this point again here to avoid misunderstandings. One may also have called the phenomenon discussed here "self-confinement", as the fields themselves "prepare" the effective potential barrier prohibiting the escape to infinity.

for the contribution from a single channel:

$$Z_{\rm ch}(t) = \int_Q^\infty \frac{dx}{\sqrt{2\pi t\hbar}} \left(1 + \frac{\hbar^2 t}{24}\frac{\partial^2}{\partial x^2} + \frac{\hbar^4 t^2}{1152}\frac{\partial^4}{\partial x^4} + \frac{\hbar^6 t^3}{82944}\frac{\partial^6}{\partial x^6} + \frac{\hbar^8 t^4}{7962624}\frac{\partial^8}{\partial x^8} + \cdots\right)\frac{1}{2\sinh\xi}, \tag{39}$$

where again $\xi = \hbar g x t/2$. The expansion parameter $\hbar^2 t/Q^2 \ll 1$ in (39) is the squared ratio of the "diffusion" length mentioned in Section 3 and the separation scale Q. After performing the differentiations and integrating over x we finally have for all four channels:

$$Z(t) = K\left[4\ln\coth\frac{\xi}{2} - \left(\frac{\lambda^2}{2^3\cdot 3}\partial_\xi + \frac{\lambda^4}{2^9\cdot 3^2}\partial_\xi^3 + \frac{\lambda^6}{2^{14}\cdot 3^4}\partial_\xi^5 + \frac{\lambda^8}{2^{21}\cdot 3^5}\partial_\xi^7\cdots\right)\frac{1}{\sinh\xi}\right]_{\xi=u} \tag{40}$$

with $u = \hbar g t Q/2$ and $\lambda^2 = g^2\hbar^4 t^3$. This expression, with the exception of the logarithm, can be written as a power series in u by substituting the Bernouilli expansion of the hyperbolic cosecans:

$$\frac{1}{\sinh\xi} = \frac{1}{\xi} - 2\sum_{n=1}^\infty \frac{2^{2n-1}-1}{(2n)!}B_{2n}\xi^{2n-1}. \tag{41}$$

where B_{2n} denotes the Bernoulli numbers ($B_2 = 1/6, B_4 = -1/30, B_6 = 1/42, B_8 = -1/30, \ldots$). It is then easy to see that there are three kinds of terms in the series involving derivatives ∂_ξ^{2n-1} in (40):

(i) terms with a positive power of u^2; there is an infinite number of such terms;

(ii) terms with an inverse power of u^2; there is one such term at for each derivative;

(iii) terms independent of u, and thus independent of Q; again there is only one such term for each derivative.

The terms of type (i) are of the order $\lambda^{2n}(\hbar Q t)^{k-4n}$, where k denotes the overall power of $\hbar$. The leading term in $Z(t)$ at the same power of $\hbar$ being of order $(\hbar Q t)^k$, arising from the expansion of the logarithmic term in (40), the contribution from the series is suppressed by a factor $(tQ^4)^{-n} \ll 1$. For the terms of type (ii) one can show that they are of the order $(\hbar Q t)^{m+1}(\hbar^4 t^3)^n$. For a given power $k = 4n - m - 1$ of $\hbar$ their ratio to the leading contribution $(\hbar Q t)^k$ is again of order $(tQ^4)^{-n} \ll 1$. Finally, for the terms of the kind (iii), their ratio to the leading terms at a given power $k = 4n$ is $(\hbar^4 t^3)^n(\hbar Q t)^{-n} = (tQ^4)^{-n} = (tQ^4)^{-k/4} \ll 1$. We thus conclude that each term arising from the series of derivatives in (40) is smaller than the contribution from the leading term, and the relative suppression increases with k or n. However, it would be wrong to conclude that these subdominant terms can all be neglected, because there are certain terms among those of type (iii), which are independent of the cutoff Q and which are not canceled by similar contributions from the channels.

Having established that the terms arising from the logarithmic term in (40) dominate in the expansion of $Z(t)$ in powers of $\hbar$, we now proceed to give those explicitly. The integrated form of the series (41) is:

$$\ln\coth\frac{u}{2} = -\ln\frac{u}{2} + \sum_{k=1}^{\infty}\frac{(2^{2k-1}-1)}{k(2k)!}B_{2k}u^{2k}, \tag{42}$$

This series converges for $u^2 < \pi^2$ or $(\hbar g t Q) < 2\pi$, which is consistent with the condition permitting the use of the Wigner representation in the square $[-Q,Q]$ and with the inequality $tQ^4 \gg 1$. We thus get for the partition function inside the four channels, up to terms involving powers of $(tQ^4)^{-1} \ll 1$:

$$Z^{[Q,\infty]}_{0+2+4+6+8}(t) = K\left[4\ln\frac{4}{\hbar g t Q} + \frac{1}{12}(\hbar g t Q)^2 - \frac{7}{2^7\cdot 3\cdot 5!!}(\hbar g t Q)^4 \right. \\ \left. + \frac{31}{2^9\cdot 3\cdot 9!!}(\hbar g t Q)^6 - \frac{127}{2^{16}\cdot 5\cdot 9!!}(\hbar g t Q)^8 + \cdots\right]. \tag{43}$$

We retained in (43) the higher-order terms for future use.

The contributions Z_0 and Z_2 for the central square are given by (24) and by eq. (5) from [7]. Their sum is:

$$Z^{[-Q,Q]}_{0+2}(t) = K\left[2\ln\sqrt{2}t^{1/2}g^2Q^2 + C - \frac{1}{12}(\hbar g t Q)^2\right]. \tag{44}$$

Adding the first two terms in (43) and (44) we finally obtain for Z up to the order of $\hbar^2$ the expression (4) found in [7], i. e. the Q-dependent corrections to the TF term vanish at the second-order of $\hbar$. Below we shall see that this statement is correct even up to the order $\hbar^8$. We conclude that the unusually large number renormalizing the TF term in (32) is an artefact of the neglect of the quantum character of the motion along the axis in the hyperbolic channels.

6. The corrections to the TF term vanish up to order $\hbar^8$

In this Section we show that the leading Q-dependent quantum corrections to the TF term up to the order $\hbar^8$ cancel if we treat the quantum mechanical motion in the channels correctly. Actually, we already gave the corrections to $Z(t)$ up to $\hbar^8$ for the correct motion in the hyperbola channels (see formula (43)). It remains for us to calculate these corrections for the square $x, y \in [-Q,Q]$, a straightforward though cumbersome task. First of all,we need to know $W_4(t)$ using (12).There are two types of contribution to $W_4(t)$: With $m-n=2$ and $m=n$. Terms with $m-n=2$ give the main contribution of the order tQ^4, terms with $m=n$ contain only logarithms of tQ^4 and may be neglected with the precision $\ln(tQ^4)/tQ^4 \ll 1$. Here we give the expression for $W_4(t)$ after the integration over p_x and p_y:

$$\int d\Gamma\, W_4\, e^{-Vt} = \frac{2\pi g^2 t^3}{2^4 t}\left(\frac{1}{5}g^2 t I_{20} - \frac{11}{45}(g^2t)^2 I_{31} \right. \\ + \frac{1}{36}(g^2t)^3 I_{42} - \frac{4}{15}I_{00} + g^2 t I_{11} \\ \left. - \frac{17}{45}(g^2t)^2 I_{22} + \frac{1}{36}(g^2t)^3 I_{33}\right) \tag{45}$$

Integration of I_{mn} over x and y for $m \neq n$ was already done before (see (25)); for $m = n$ we obtain:

$$\begin{aligned}
I_{00} &= \frac{\sqrt{2\pi}}{(g^2t)^{1/2}}\left[\ln(g^2Q^4t) + C + \ln 2\right] \\
I_{11} &= \frac{\sqrt{2\pi}}{(g^2t)^{3/2}}\left[\ln(g^2Q^4t) + C + \ln 2 - 2\right] \\
I_{22} &= \frac{3\sqrt{2\pi}}{(g^2t)^{5/2}}\left[\ln(g^2Q^4t) + C + \ln 2 - \frac{8}{3}\right] \\
I_{33} &= \frac{15\sqrt{2\pi}}{(g^2t)^{7/2}}\left[\ln(g^2Q^4t) + C + \ln 2 - \frac{46}{15}\right].
\end{aligned} \tag{46}$$

Collecting all terms together we have from (45) for the corrections to $Z(t)$ of the order of $\hbar^4$ in the square $x, y \in [-Q, +Q]$:

$$\begin{aligned}
Z_4^{[-Q,Q]}(t) = K\Big[&\frac{7}{2^7 \cdot 3 \cdot 5!!}g^4\hbar^4t^4Q^4 \\
&+g^2\hbar^4t^3[\ln(g^2Q^2\sqrt{t/2}) - 8C - 16\ln 2]\Big]
\end{aligned} \tag{47}$$

where the first term comes from I_{mn} with $m - n = 2$, and the logarithmic terms, as we remarked before, arise from the contributions I_{mm}. We see that the second term in (47) is of order $\ln(tQ^4)/Q^4t \ll 1$. Discarding it and adding the $\hbar^4$ correction from the channels in (43) we find that with the precision $\ln(tQ^4)/tQ^4 \ll 1$,

$$Z_4^{[-Q,+Q]} + Z_4^{[Q,\infty]} = 0 \tag{48}$$

as it was for the second-order corrections (with the precision $1/tQ^4$).

Consider now the $\hbar^6$-order corrections to the partition function. In (43) we included the dominant contribution from the channels $[Q, \infty]$ (fourth term). For W_6 in the square we have, after integration over p_x and p_y and using the notation of (22):

$$\begin{aligned}
\int d\Gamma\, W_6\, e^{-tV} = \frac{2\pi(gt)^6}{t \cdot 9!! \cdot 2^6}\Big[&-61I_{30} + \frac{249}{2}(g^2t)I_{41} \\
&-\frac{119}{4}(g^2t)^2I_{52} + \frac{35}{24}(g^2t)^3I_{63}\Big],
\end{aligned} \tag{49}$$

Using (25) for the evalulation of the I_{mn} and collecting all terms, we obtain for $Z_6^{[-Q,Q]}$ in the square:

$$Z_6^{[-Q,Q]}(t) = -K\frac{31}{3 \cdot 2^9 \cdot 9!!}(\hbar g t Q)^6. \tag{50}$$

Once more, the contribution from the square is exactly canceled against the one from the four channels given in (43), as it happened for the second and fourth order:

$$Z_6^{[-Q,Q]}(t) + Z_6^{[Q,\infty]}(t) = 0 \tag{51}$$

with the precision $\ln(tQ^4)/(tQ^4) \ll 1$.

Finally, we give here the order $\hbar^8$ corrections to $Z_8^{[-Q,Q]}$ in the square:

$$\begin{aligned}
\int d\Gamma\, W_8\, e^{-tV} = \frac{2\pi(gt)^8}{t \cdot 5!! \cdot 2^7}\Big[&\frac{1261}{7560}I_{40} - \frac{259}{540}g^2tI_{51} + \frac{893}{5040}(g^2t)^2I_{62} \\
&-\frac{23}{1296}(g^2t)^3I_{73} + \frac{5}{10368}(g^2t)^4I_{84}\Big].
\end{aligned} \tag{52}$$

where we again discarded terms proportional to I_{mm}, which are of the order $\ln(tQ^4)/(tQ^4) \ll 1$ with respect to the terms containing I_{mn} with $m > n$. Using now the general expression (25) for I_{mn} we obtain:

$$\int d\Gamma\, W_8\, e^{-Vt} = \frac{(2\pi)^{3/2} 127}{g t^{3/2} 2^9 \cdot 5!! \cdot 3 \cdot 2^7 \cdot 7!!} (gtQ)^8 \tag{53}$$

and get the result

$$Z_8^{[-Q,Q]}(t) = K \frac{127}{5 \cdot 2^{16} \cdot 9!!} (\hbar g t Q)^8 \tag{54}$$

which miraculously cancels with the contribution from the channels in (43).

Although we cannot prove such a cancellation in general, to all orders, we have no doubt that all higher-order corrections to the partition function containing powers of $(\hbar g t Q)$ cancel in the limit $tQ^4 \gg 1$. Of course, there are other corrections involving powers of $(tQ^4)^{-1}$, but these are suppressed due to the classical condition of adiabaticity $tQ^4 \gg 1$. We shall consider this issue in Section 7.

In anticipation of later applications, we remark here that there is a strong correlation between the power of $\hbar$, denoted by $k \geq 2$, and power of the dominant terms in the limit $tQ^4 \gg 1$. A systematic analysis of the higher-order corrections using *Mathematica* leads to the conclusion that the difference between m and n in the leading integral I_{mn} from (22) is $m - n = \frac{1}{2}k$. Besides the terms with the largest difference $m - n$, which give the leading contribution to W_k and which we retain in (49) and (52), there are also terms involving I_{mn} with $m - n = \frac{1}{2}k - 2\ell$ with $\ell = 1, 2, \ldots < \frac{1}{4}k$. There is also a correlation between the powers of t and m, n. The analysis shows that for the terms with $m - n = \frac{1}{2}k$ the power of t is $2m - n - 1$, and for terms with $m - n = \frac{1}{2}k - 2\ell$ the power is $2m - n - 1 - 3\ell$. The factor of g for I_{mn} is g^{2m}. Straightforward calculations similar to the above show that the ratio of the contributions to the partition function for the terms $m - n = \frac{1}{2}k - 2\ell$ to the ones for $m - n = k/2$ is of the order $1/(tQ^4)^\ell \ll 1$. Indeed, we find:

$$Z_k^{(m,n)}(t) = K (\hbar g t Q)^k \frac{(2n-1)!!}{2^{n-1}} \qquad \left(m - n = \frac{k}{2} \right); \tag{55}$$

$$Z_k^{(m,n)}(t) = K (g^4 t Q^4)^{-\ell} (\hbar g t Q)^k \frac{(2n-1)!!}{2^{n-1}(k - 4\ell)} \qquad \left(m - n = \frac{k}{2} - 2\ell \right).$$

This justifies neglecting these subdominant terms in our calculations here.

Finally, at any power k of $\hbar$ terms with logarithms ($m = n$) may be discarded with respect to the terms with powers of Q^2. Indeed, the last terms with $m > n$ are of the order of $(\hbar g t Q)^k$, terms with $m = n$ are of the order of $(g^2 \hbar^4 t^3)^{k/4} \ln(t g^4 Q^4)$ and may be neglected with the precision $\ln(g^4 t Q^4)/(g^2 t Q^4) \ll 1$. Again, with increasing k the precision improves.

7. Q-independent terms in the channels and central region

We already mentioned that the derivative expansion for the channel contribution (40) has one Q-independent term for each power of of $\lambda^2 = g^2 \hbar^4 t^3$. From the expansion (41)

it is evident that these terms have the form

$$\frac{2^{2n-1}-1}{n(2n)!}(2n-1)!!B_{2n}\lambda^{2n}. \tag{56}$$

Each terms is of the order $(tQ^4)^{-1} \ll 1$ compared with the leading term from the expansion of the logarithic term in (40). Collecting these terms we obtain a series of Q-independent terms contributing to $Z(t)$:

$$\frac{1}{2^3\cdot 3}B_2\lambda^2+\frac{2^3-1}{2^{10}\cdot 3^2}B_4\lambda^4+\frac{2^5-1}{2^{14}\cdot 3^5}B_6\lambda^6+\frac{2^7-1}{2^{23}\cdot 3^5}B_8\lambda^8+\cdots \tag{57}$$

We can rewrite this series in the form

$$\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n-1}-1)(2n-1)!!}{2^{2(n-1)}n(2n)!}B_{2n}\left(\frac{\lambda}{4\sqrt{3}}\right)^{2n}. \tag{58}$$

The factor $(2n-1)!!$, which arises from the $(2n-1)$-th derivative in (40), spoils the convergence of this series for $Z(t)$. Using the identity $2^n n!(2n-1)!! = (2n)!$, we obtain for the series of Q-independent terms:

$$Z^{[Q,\infty]} = K\sum_{n=1}^{\infty}\frac{4(2^{2n-1}-1)}{2^n n!n}B_{2n}\left(\frac{\lambda}{4\sqrt{3}}\right)^{2n}. \tag{59}$$

This series is only asymptotic despite the smallness of λ^2, because the Bernouilli numbers grow factorially at large n:

$$B_{2n}\to\frac{2(2n)!}{(2\pi)^{2n}}. \tag{60}$$

Next we turn to the Q-independent contributions to $Z(t)$ from the central region. We start with the investigation of the corrections to the dominant terms. A straightforward calculation yields the following correction factor to I_{mn} from (25):

$$1-\frac{(2m-1)!!}{(2n-1)!!}(g^2Q^4t)^{n-m}. \tag{61}$$

This factor plays an important part making the corrected I_{mn} well behaved in the limit $m=n$. Indeed, we note that with $\varepsilon = 2(m-n)$:

$$\frac{(2m-1)!!}{(2n-1)!!}\approx 1+\varepsilon\sum_{\ell=1}^{m}\frac{1}{2\ell-1} \tag{62}$$

the diagonal elements $(m=n)$ become

$$I_{mm}=\lim_{\varepsilon\to 0}I_{mn}=\frac{\sqrt{2\pi}(2m-1)!!}{(g^2t)^{m+\frac{1}{2}}}\left[\ln(g^2Q^4t)-2\sum_{\ell=1}^{m}\frac{1}{2\ell-1}\right], \tag{63}$$

demonstrating that the corrections to I_{mm} are independent of Q, with the exception of the logarithmic term. One easily confirms this general result by explicit calculations. Substituting $x=w$ and $y=\sqrt{2/t}(u/gw)$ in (22) we get:

$$I_{mm}=4\left(\frac{2}{g^2t}\right)^{\frac{2m+1}{2}}\int_0^Q\frac{dw}{w}\int_0^w du\,u^{2m}e^{-u^2}. \tag{64}$$

We may write this expression as a derivative of the error function:

$$I_{mm} = 4\left(\frac{2}{g^2 t}\right)^{\frac{2m+1}{2}} \frac{\sqrt{\pi}}{2}(-1)^m \frac{d^m}{da^m}\int_0^{w_0} \frac{dw}{\sqrt{a}w}\mathrm{erf}(\sqrt{a}w)\Big|_{a=1}, \tag{65}$$

where $w_0^2 = g^2Q^4t/2$. Integrating by parts, discarding the exponentially small terms in w_0, then differentiating with respect to a and finally setting $a = 1$, we obtain:

$$I_{mm} = \frac{\sqrt{2\pi}}{(g^2t)^{m+\frac{1}{2}}}\left[(2m-1)!!\ln w_0^2 - \frac{2^m}{\sqrt{\pi}}\int_0^\infty dt\, t^{m-\frac{1}{2}}e^{-t}\ln t\right]. \tag{66}$$

For the integral in the second term of this expression one finds (see ref. [16] formula 4.352.2):

$$\int_0^\infty dt\, t^{m-\frac{1}{2}}e^{-t}\ln t = \frac{\sqrt{\pi}}{2^m}(2m-1)!!\left[2\sum_{\ell=1}^m \frac{1}{2\ell-1} - C - 2\ln 2\right] \tag{67}$$

which yields

$$I_{mm} = \frac{\sqrt{2\pi}(2m-1)!!}{(g^2t)^{m+\frac{1}{2}}}\left[\ln(g^2Q^4t) + C + \ln 2 - 2\sum_{\ell=1}^m \frac{1}{2\ell-1}\right]. \tag{68}$$

Next we use a trick, first introduced by Euler, replacing the finite sum by an asymptotic series (see e.g. [19]):

$$\sum_{\ell=1}^m \frac{1}{2\ell-1} = \frac{1}{2}\left[C + \ln(2m) + \sum_{\ell=1}^\infty \frac{2^{2\ell-1}-1}{(8m^2)^\ell}B_{2\ell}\right]. \tag{69}$$

We are now ready to combine the contribution from the central region with the asymptotic series (59) obtained earlier for the Q-independent contribution for the channels. A special case is the case $m = n = 0$, for which there is no infinite sum, giving

$$I_{00} = \sqrt{\frac{2\pi}{g^2t}}\left[\ln(g^2Q^4t) + C + \ln 2\right]. \tag{70}$$

Taken together with the logarithmic term from the channels, this leads to the Q-independent expression (4) obtained in the improved TF approximation [7]. Our detailed analysis has shown that the structures I_{mm} containing Q-independent terms appear only in W_{4k} with $k = 1, 2, \ldots$. This means that such terms appear only at the orders $\hbar^{4k}$ and implies that the expansion parameter of $Z(t)$ is $\lambda^2 = g^2\hbar^4t^3$.

For a given power λ^{2n} there are $3n$ quantities I_{mm} ($m = 1, 2, \ldots, 3n$) and I_{00}, which we collect separately. The results given above allow us to write the Q-independent asymptotic series from the central region. Including also the TF term (4), we thus obtain:

$$\begin{aligned} Z^{[-Q,Q]}(t) = K\Bigg[&\ln\frac{1}{g^2\hbar^4t^3} + 9\ln 2 + C \\ &+ \sum_{n=1}^\infty \lambda^{2n}\Big(a_0^{(n)}(C+\ln 2) \\ &- \sum_{m=1}^{3n} a_m^{(n)}(2m-1)!!\left[\ln m + \sum_{\ell=1}^\infty \frac{2^{2\ell-1}-1}{(8m^2)^\ell}B_{2\ell}\right]\Big)\Bigg]. \end{aligned} \tag{71}$$

Here the $a_m^{(n)}$ $(m = 0, 1, 2, \ldots, 3n)$ denote the numerical coefficients in $(g^2 t)^m I_{mm}$ occurring in Z_{4n}. For example, for $n = 1$ we have four such numbers with alternating signs:

$$a_0^{(1)} = -\frac{1}{60}, \quad a_1^{(1)} = \frac{1}{16}, \quad a_2^{(1)} = -\frac{17}{720}, \quad a_3^{(1)} = \frac{1}{576}. \tag{72}$$

Summing up all these results, we may surmise that in the limit $tQ^4 \gg 1$ all Q-dependence is canceled, and we are left with two asymptotic series (59) and (71) from the two regions contributing together to the partition function of the two-dimensional YMQM model. We have shown explicitly how the small parameter $(tQ^4)^{-1}$, artificially introduced in the adiabatic separation of the degrees of freedom in the channels, transmutes into the genuine quantum mechanical parameter $\lambda^2 = g^2\hbar^4 t^3$.

8. Conclusions

We have shown that the richness of the classical YM mechanics with a x^2y^2 potential translates into, and even gets amplified by, the quantum mechanical properties of the system. The YM quantum mechanics exhibits a confinement property, which strongly influences the quantum mechanical motion in the x^2y^2 potential. At higher order in $\hbar$ (up to $\hbar^8$) this results in the vanishing of the leading quantum corrections (for $tQ^4 \gg 1$), when we correctly take into account this property for the motion in the hyperbolic channels. We believe that this result may survive to even higher order. We also derived a novel form of the equation for the Uhlenbeck-Beth function $W(\vec{r}, \vec{p}; t)$, which is the basis of the WK expansion. We believe that our expression can be a starting point for new approximation schemes using techniques from diffusion theory. We hope that the lessons derived from the present study of the higher-order quantum corrections to the homogeneous limit of the Yang-Mills equations will be useful for an improved understanding of the internal dynamics of the Yang-Mills quantum field theory.

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